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SOLUTION OF PROBLEMS IN NUMBER TWO.

SOLUTIONS of problems in No. 2 have been received as follows:

From Marcus Baker, 107; G. M. Day, 107; Prof. A. B. Evans, 107, 108, 109 and 110; E. S. Farrow, 107; Henry Gunder, 107 and 109; H. Heaton, 107, 108 and 110; Prof. A. Hall, 109; Prof. Del. Kemper, 107 and 109; Christine Ladd, 107; Prof. H. T. J. Ludwick, 109; Artemas Martin, 107, 108 and 109; Dr. A. B. Nelson, 107 and 109; L. Regan, 107; E. B. Seitz, 107, 108 and 110.

We omitted, by oversight in No. 2, to credit Dr. Eggers with a solution of problem 102.

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107. "If $c = \sqrt{(a^2 - b^2)}$, show that

$$\frac{a+c}{a-c} = \left(\frac{c+a-b}{c-a+b}\right)^2."$$

SOLUTION BY CHRISTINE LADD, UNION SPRINGS, N. Y.

IF we put $m = a + c$, $n = a - c$, the equation becomes

$$\frac{m}{n} = \left(\frac{m-b}{b-n}\right)^2.$$

Hence $$m(b^2 - 2bn + n^2) = n(m - 2mb + b^2),$$

$$b^2 = \frac{nm^2 - mn^2}{m-n} = mn = a^2 - c^2, \text{ whence } c = \sqrt{(a^2 - b^2)}.$$

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108. "Employing the notation of Prob. 100, page 31, show by Finite Differences, or otherwise, that

$$\frac{1}{a_x} \pm \frac{1}{a} = \frac{1}{b_x} \pm \frac{1}{b} = \frac{1}{c_x} \pm \frac{1}{c} = \frac{1}{6}\left([5+2\sqrt{6}]^x + [5-2\sqrt{6}]^x \pm 2\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$
$$+ \frac{1}{2\sqrt{6}}\left([5+2\sqrt{6}]^x - [5-2\sqrt{6}]^x\right)\left(\sqrt{\left[\frac{2}{ab}+\frac{2}{ac}+\frac{2}{bc}\right]}\right);$$ where the double sign is to be taken *plus* when x is odd and *minus* when x is even."

SOLUTION BY PROF. A. B. EVANS, LOCKPORT, N. Y.

LET $\frac{1}{a_x} + \frac{1}{b_x} + \frac{1}{c_x} = u_x$ and $\left(\frac{2}{a_x b_x} + \frac{2}{a_x c_x} + \frac{2}{b_x c_x}\right)^{\frac{1}{2}} = v_x$; . . . (1)

then $\frac{1}{a_x} = -\frac{1}{a_{x-1}} + 2u_{x-1} + 2v_{x-1}$, $\frac{1}{b_x} = -\frac{1}{b_{x-1}} + 2u_{x-1} + 2v_{x-1}$,

$$\frac{1}{c_x} = -\frac{1}{c_{x-1}} + 2u_{x-1} + 2v_{x-1}. \quad . \; . \; . \; . \; (2)$$

From (1) when $x = 0$, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = u$ and $\left(\frac{2}{ab}+\frac{2}{ac}+\frac{2}{bc}\right)^{\frac{1}{2}} = v; \dots$ (3)

and from (2) when $x = 1$,

$$\frac{1}{a_1} = -\frac{1}{a} + 2u + 2v,\ \frac{1}{b_1} = -\frac{1}{b} + 2u + 2v,\ \frac{1}{c_1} = -\frac{1}{c} + 2u + 2v \dots (4)$$

From (4), $$\frac{1}{a_1} + \frac{1}{b_1} + \frac{1}{c_1} = u_1 = 5u + 6v, \dots\dots\dots (5)$$

and $$\left(\frac{2}{a_1b_1} + \frac{2}{a_1c_1} + \frac{2}{b_1c_2}\right)^{\frac{1}{2}} = v_1 = 4u + 5v. \dots\dots\dots (6)$$

From (2), $u_x = 5u_{x-1} + 6v_{x-1}$ and $v_x = 4u_{x-1} + 5v_{x-1}. \dots\dots$ (7)

By giving x integral values from 1 to x inclusive, we may obtain a series of equations from (2) that will enable us to find readily

$$\frac{1}{a_x}\pm\frac{1}{a} = \frac{1}{b_x}\pm\frac{1}{b} = \frac{1}{c_x}\pm\frac{1}{c} = 2[(u_{x-1}+v_{x-1})-(u_{x-2}+v_{x-2})+(u_{x-3}+v_{x-3}) - \dots \pm (u+v)]; \dots (8)$$

where the double sign is to be taken + when x is odd and — when x is even.

From the first of equations (7) $v_{x-1} = \frac{1}{6}(u_x - 5u_{x-1})$, and therefore $v_x = \frac{1}{6}(u_{x+1} - 5u_x)$. These values of v_{x-1} and v_x, substituted in the second of equations (7), give $u_{x+1}-10u_x+u_{x-1} = 0$. $\therefore u_x-10u_{x-1}+u_{x-2}=0$. (9)

Equation (9) is an equation in finite differences whose solution is

$$u_x = C_1(r_1)^x + C_2(r_2)^x; \dots\dots\dots\dots (10)$$

where $r_1 = 5 + 2\sqrt{6}$ and $r_2 = 5 - 2\sqrt{6}$ are the roots of $x^2-10x+1=0$, and C_1, C_2 are constants of integration. To determine these constants, we observe that when $x=0$ and $x=1$, equation (10) gives $u = C_1 + C_2$ and $u_1 = 5u + 6v = C_1r_1 + C_2r_2$; and therefore $C_1 = \frac{1}{2}u + \frac{1}{4}v\sqrt{6}$ and

$$C_2 = \tfrac{1}{2}u - \tfrac{1}{4}v\sqrt{6}. \dots\dots\dots\dots (11)$$

Again, since $v_x = \frac{1}{6}(u_{x+1} - 5u_x)$, we have

$$u_x + v_x = \tfrac{1}{6}(u_{x+1} + u_x) = \tfrac{1}{6}C_1(r_1^{x+1}+r_1^x) + \tfrac{1}{6}C_2(r_2^{x+1} + r_2^x) \dots (12)$$

Observing that $r_1 + 1 = 6 + 2\sqrt{6}$ and $r_2 + 1 = 6 - 2\sqrt{6}$ we may write (12)

$$u_x + v_x = \tfrac{1}{6}(6 + 2\sqrt{6})C_1(r_1)^x + \tfrac{1}{6}(6 - 2\sqrt{6})C_2(r_2)^x. \dots (13)$$

From (8) and (13) we find $\frac{1}{a_x}\pm\frac{1}{a} = \frac{1}{b_x}\pm\frac{1}{b} = \frac{1}{c_x}\pm\frac{1}{c}$

$$= \tfrac{1}{3}(6+2\sqrt{6})C_1[r_1^{x-1}-r_1^{x-2}+r_1^{x-3}\dots\pm1]+\tfrac{1}{3}(6-2\sqrt{6})C_2[r_2^{x-1}-r_2^{x-2}\dots\pm1]$$

$$= \frac{6+2\sqrt{6}}{3(r_1+1)}C_1\left(r_1^x\pm1\right)+\frac{6-2\sqrt{6}}{3(r_2+1)}C_2\left(r_2^x\pm1\right) = \tfrac{1}{3}C_1(r_1^x\pm1)+\tfrac{1}{3}C_2(r_2^x\pm1)$$

$$= \frac{u}{6}\left[(5+2\sqrt{6})^x+(5-2\sqrt{6})^x\pm2\right]+\frac{v}{2\sqrt{6}}\left[(5+2\sqrt{6})^x-(5-2\sqrt{6})^x\right]$$

$$= \frac{1}{6}\left[(5+2\sqrt{6})^x+(5-2\sqrt{6})^x\pm2\right]\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

$$+ \frac{1}{2\sqrt{6}}\left[(5+2\sqrt{6})^x-(5-2\sqrt{6})^x\right]\left(\frac{2}{ab}+\frac{2}{ac}+\frac{2}{bc}\right)^{\frac{1}{2}}.$$

109. "Show that the determinant

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$$

is divisible by $(a+b)^2-(c+d)^2$; and by $(a-b)^2-(c-d)^2$."

SOLUTION BY HENRY GUNDER, NORTH MANCHESTER, IND.

ADDING the 2nd, 3rd and 4th columns to the first, we have

$$(a+b+c+d)\begin{vmatrix} 1 & b & c & d \\ 1 & a & d & c \\ 1 & d & a & b \\ 1 & c & b & a \end{vmatrix}.$$

Subtracting the 3rd and 4th, and adding the 2nd column to the 1st, we get

$$(a+b-c-d)\begin{vmatrix} 1 & b & c & d \\ 1 & a & d & c \\ -1 & d & a & b \\ -1 & c & b & a \end{vmatrix}.$$

This shows that the given determinant is divisible by $(a+b+c+d)$ and by $(a+b-c-d)$, and hence by their product $(a+b)^2-(c+d)^2$.

By adding the 3rd column and subtracting the 2nd and 4th, we get the factor $(a-b+c-d)$. Subtracting the 2nd and 3rd and adding the 4th column we get the factor $(a-b-c+d)$. Therefore the determinant is divisible by $(a-b)^2-(c-d)^2$. (See ANALYST, VOL. III, No. 1.)

[This question was solved in a manner similar to the above by D. J. Mc Adam, Dr. Nelson, Prof. Kemper and Prof. Hall.]

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

BY reduction the value of the determinant is found to be

$$(a^2+b^2)^2-(a-b)^2(c+d)^2-(a+b)^2(c-d)^2+(c^2-d^2)^2,$$
$$=[(a+b)^2-(c+d)^2][(a-b)^2-(c-d)^2].$$

[Prof. Ludwick and Prof. Evans solved this question in the same manner as Mr. Martin.]

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110. "A circle, radius r, is placed at random on another equal circle. Prove that the average area of the greatest ellipse that can be inscribed in the area common to the two circles, is $\pi r^2(\frac{1}{2}\pi-\frac{4}{3})$."

SOLUTION BY HENRY HEATON, DES MOINES, IOWA.

LET $2a$ = the distance between the centers of the circles. The distance between the center of the ellipse and that of one of the circles will $= a$, and the equation of the ellipse, taking the center of one of the circles as the origin, is $B^2(x-a)^2 + A^2y^2 = A^2B^2$;

whence $$\frac{dy}{dx} = \frac{-B^2(x-a)}{A^2y}.$$

The equation of the circle is $x^2+y^2=r^2$.

$$\therefore \quad \frac{dy}{dx} = -\frac{x}{y}.$$

At the point where the curves are tangent, x, y, and $dy \div dx$ of the ellipse and x, y, and $dy \div dx$ of the circle coincide. Combining the above equations we get $A = \sqrt{\left[\left(r^2-ax\right)\left(\frac{x-a}{x}\right)\right]}$, and $B = \sqrt{(r^2-ax)}$.

$\therefore \; AB = \left(r^2-ax\right)\sqrt{\left(\frac{x-a}{x}\right)}$. $\therefore \; \frac{d(AB)}{dx} = \frac{a}{2x^2}\left(r^2-ax\right)\sqrt{\left(\frac{x}{x-a}\right)}$ $- a\sqrt{\left(\frac{x-a}{x}\right)}$. Putting this differential coefficient $=0$ to find the value of x that will make AB a maximum, we get $x = \frac{1}{4}[\sqrt{(8r^2+a^2)}+a]$, or

$$a = \frac{2x^2-r^2}{x}. \quad \therefore \; da = \left(\frac{2x^2+r^2}{x^2}\right)dx.$$

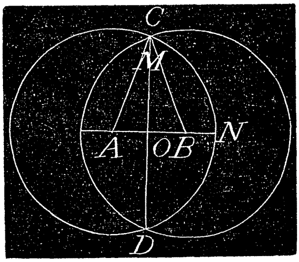
The mean area of the ellipse $= \frac{1}{\pi r^2}\int_0^r \pi AB \times 2\pi a da = \frac{2\pi}{r}\int_0^r ABada$

$$= \frac{4\pi}{r^2}\int_{\frac{1}{2}r\sqrt{2}}^{r} \frac{4x^4-r^4}{x^4} \times (r^2-x^2)^{\frac{3}{2}}dx = \frac{16\pi}{r^2}\int (r^2-x^2)^{\frac{3}{2}}dx$$

$$-4\pi r^2\int x^{-4}(r^2-x^2)^{\frac{3}{2}}dx = \pi r^2\left(\frac{\pi}{2}-\frac{4}{3}\right).$$

SOLUTION BY E. B. SEITZ, GREENVILLE, OHIO.

LET A and B be the centers of the two circles, CD the common chord, OM and ON the semi-axes of the greatest ellipse that can be inscribed in the area common to the two circles.



Put $AC=r$, $AB=z$, $OM=x$, $ON=y$. Then if we represent the coordinates by m, n, the equation to the circle whose center is A, referred to AB and CD, is $(m+\frac{1}{2}z)^2+n^2=r^2$, . . (1) and the eqn. to the ellipse is $x^2m^2+y^2n^2=x^2y^2$. (2)

From (1) and (2) by eliminating n, we have

$$(x^2-y^2)m^2 - y^2zm + y^2(r^2-x^2-\tfrac{1}{4}z^2) = 0. \; . \; . \; . \; . \; (3)$$

But, since the ellipse is tangent to the circle, the two values of m in (3) must be equal; hence we have

$$(x^2-y^2)(4r^2-4x^2-z^2) = y^2z^2. \; . \; . \; . \; . \; . \; . \; . \; . \; (4)$$

From (4) we find $x^2y^2 = x^4 - \dfrac{x^4z^2}{4(r^2 - x^2)} =$ a maximum. . . . . (5)

Differentiating (5), regarding z as constant, equating to zero, and reducing, we have $(2r^2 - x^2)z^2 = 8(r^2 - x^2)^2$. . . . . (6)

From (6) we have $z^2 = \dfrac{8(r^2 - x^2)^2}{2r^2 - x^2}$. . . . . (7)

Substituting the value of z^2 in (5), reducing, and extracting the square root, we have $xy = \dfrac{x^3}{(2r^2 - x^2)^{1/2}}$. . . . . (8)

Differentiating (7), we have $zdz = \dfrac{8r^4xdx}{(2r^2 - x^2)^2} - 8xdx$. . . . . (9)

If A denotes the average area of the ellipse, we have

$$A = \frac{\int_0^{2r} \pi xy.2\pi zdz}{\int_0^{2r} 2\pi zdz} = \frac{\pi}{2r^2}\int_0^{2r} xyzdz. \;.\;.\;.\;.\;.\;. \quad (10)$$

Substituting in (10) from (8) and (9), and observing that when $z = 0$, $x = r$, and when $z = 2r$, $x = 0$, we have

$$A = \frac{4\pi}{r^2}\int_0^r \frac{x^4dx}{(2r^2 - x^2)^{1/2}} - 4\pi r^2 \int_0^r \frac{x^4dx}{(2r^2 - x^2)^{\frac{5}{2}}}. \;.\;.\;.\;.\;. (11)$$

Putting $\sin\theta = x \div r\sqrt{2}$, (11) becomes

$$A = 16\pi r^2 \int_0^{\frac{1}{4}\pi} \sin^4\theta d\theta - 4\pi r^2 \int_0^{\frac{1}{4}\pi} \tan^4\theta d\theta = \pi r^2\left(\frac{\pi}{2} - \frac{4}{3}\right).$$

[Prob. 89. Though we have already published two solutions of this problem, yet as both are in some respects confessedly defective, and as the method adopted in the following solution differs from both the published solutions, we add another solution of 89.

In justice to Mr. Siverly and to Mr. Adcock, we take pleasure in stating that Mr. Siverly corrected his statement that the spheres would roll in opposite directions, several months ago, and by rigorous analysis had arrived at the same conclusion, in relation to the rolling of the lower sphere, as that deduced in the following solution by Mr. Hill. And, since the issue of No. 2, we have received from Mr. Adcock a revision of his published solution, in which, after correcting equations (6) and (9), he obtains an equation for the required curve which corresponds with that given by Mr. Siverly, and consequently with the following by Mr. Hill. And he also arrives at the same conclusion as Mr. Siverly and Mr. Hill, in relation to the rolling of the lower sphere.